# Real Closed Field

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We synthesise material from [1], [4], [2] and [3] as convenient.

Note that the meaning of the notation $F(\sqrt{a})$ depends on whether $a$ is a square in $F$. We take care to write it only in the non-trivial case. The same applies to $F(i)$.

## 0.1 Ordered Fields

We begin with a purely algebraic characterisation of an ordered field. This relies on the theory of *ring orderings*, which can be found in [5].

**Theorem 1.** *A field can be ordered if and only if it is real (that is, $-1$ is not a sum of squares).*

*Proof.* $\square$

We can also characterise algebraically whether a field $F$ has a unique ordering. This allows us to talk about 'the ordering' on $F$ without ambiguity.

**Lemma 2.** *Let $F$ be a real field. There is a unique ordering on $F$ if and only if, for each $a \in F$, either $a$ or $-a$ is a sum of squares. In this ordering, the non-negative elements are precisely the squares in $F$.*

*Proof.* $\square$

**Corollary 3.** *Let $F$ be an ordered field. Then $F$ has a unique field ordering if and only if every non-negative element is a sum of squares.*

*Proof.* $\square$

**Corollary 4.** *There is a unique field ordering on $\mathbb{Q}$.*

*Proof.* We use Corollary 3. Let $x \in \mathbb{Q}$ be non-negative. Then $x = p/q$ for some integers $p \geq 0$ and $q > 0$, so

$$x = \underbrace{\frac{1}{q^2} + \cdots + \frac{1}{q^2}}_{pq \text{ times}}.$$

$\square$

There is a corresponding algebraic characterisation of an ordered field extension.

**Theorem 5.** *Let $F$ be an ordered field, and let $K/F$ be a field extension. Then there is an ordering on $K$ making the extension $K/F$ ordered if and only if $\sum_i x_i \alpha_i^2 \neq -1$ for all choices of $\alpha_i \in K$ and $x_i \in F_{\geq 0}$.*

*Proof.* $\square$

**Lemma 6.** *Let $F$ be an ordered field, and let $K/F$ be a field extension. Suppose that there is an $F$-linear functional $\pi : K \to F$ such that, for all $x \in K$, $\pi(x^2) \geq 0$. Then there is a field ordering on $K$ making $K/F$ ordered.*

*Proof.* Consider the sum $\sum_i x_i \alpha_i^2$ for some $\alpha_i \in K$ and $x_i \in F_{\geq 0}$. By $F$-linearity, we compute

$$\pi\left(\sum_i x_i \alpha_i^2\right) = \sum_i x_i \pi(\alpha_i^2) \geq 0.$$

Since $\pi(-1) = -1 < 0$, we are done by Theorem 5. $\square$

**Corollary 7.** *Let $F$ be a ordered field, and suppose $a \in F$ is non-negative (and not a square). Then there is a field ordering on $F(\sqrt{a})$ making $F(\sqrt{a})/F$ ordered.*

*Proof.* Let $\pi : F(\sqrt{a}) \to F$ be the projection induced by the $F$-basis $\{1, \sqrt{a}\}$. For $x, y \in F$, we have $\pi((x + y\sqrt{a})^2) = x^2 + ay^2 \geq 0$. We are done by Lemma 6. $\square$

**Lemma 8.** *Let $F$ be an ordered field, and let $K/F$ be an odd-degree extension. Then there is a field ordering on $K$ making $K/F$ ordered.*

*Proof.* Since $\operatorname{char} R = 0$, we can apply the primitive element theorem. Let $K = F(\alpha)$ for some $\alpha \in K$, and let $f$ be the minimal polynomial of $\alpha$ over $K$. Then $\deg f = [K : F]$ is odd. By Theorem 5, we need to show that the congruence

$$\sum_i a_i g_i^2 \equiv -1 \bmod f \tag{$\star$}$$

fails to hold for any non-negative $a_i \in F$ and polynomials $g_i \in F[X]$ each of degree at most $\deg f - 1$. Proceed by induction on $\deg f$; if $\deg f = 1$, then ($\star$) reduces to an equality of a non-negative element of $F$ with a negative one. Otherwise, suppose for a contradiction that ($\star$) holds. Without loss of generality, we may assume, for all $i$, we have $a_i \neq 0$ and that $\deg g_i < \deg f$.

Rearranging ($\star$), we have $\sum_i a_i g_i^2 + 1 = hf$ for some $h \in F[X]$. Let $d = \max_i \deg g_i$; note that $d < \deg f$ by construction. Since each $a_i$ is positive, the $2d$th coefficient on the left-hand side must be positive. Therefore

$$\deg h + \deg f = \deg\left(\sum_i a_i g_i^2 + 1\right) = 2d.$$

Then $\deg h$ is odd, so $h$ has an odd-degree irreducible factor $\tilde{h}$. We have

$$\deg \tilde{h} \leq \deg h = 2d - \deg f < \deg f,$$

but $\sum_i a_i g_i^2 \equiv -1 \bmod \tilde{h}$. We are done by induction.$_\#$ $\square$

There is an easier way to construct ordered field extensions if we don't care about them being algebraic.

**Lemma 9.** *Let $F$ be an ordered field, and let $a \in F$. Then there is a unique ordering on the function field $F(X)$ making $F(X)/F$ ordered such that $X > a$ but $b > X$ for $b > a$, and a unique one such that $X < a$ but $b < X$ for $b < a$.*

Intuitively, $X$ is infinitesimally close to $a$. When $a = 0$, we often write $R(\varepsilon)$ for $R(X)$ with the first type of ordering.

## 0.2 Real Closed Fields

**Definition 10.** A real closed field $F$ is a real field such that

1. for all $x$ in $F$, either $x$ or $-x$ is a square in $F$, and
2. every odd-degree polynomial over $F$ has a root in $F$.

**Lemma 11.** *An field $F$ is real closed iff*

1. *for all nonzero $x$ in $F$, exactly one of $x$ and $-x$ is a square, and*
2. *every odd-degree polynomial over $F$ has a root in $F$.*

*Proof.* For the reverse direction, exactly one of 1 and $-1$ is a square, so $-1$ is not a square. $\square$

**Lemma 12.** *An ordered field is real closed iff*

1. *every non-negative element is a square, and*
2. *every odd-degree polynomial over $F$ has a root in $F$.*

*Proof.* For the reverse direction, squares in an ordered field are non-negative. $\square$

Fix a real closed field $R$.

**Lemma 13.** *$R$ has a unique ordering making it an ordered field. In this ordering, the non-negative elements are exactly the squares.*

*Proof.* This is Lemma 2. $\square$

In what follows, all algebraic extensions are given up to $R$-isomorphism, as is conventional. Observe that, since $-1$ is not a square in $R$, $R(i)/R$ is a quadratic extension. We will show this is the **only** nontrivial algebraic extension of $R$.

**Lemma 14.** *There is no nontrivial odd-degree finite extension of $R$.*

*Proof.* Let $K/R$ be an odd-degree extension of $R$. By the primitive element theorem, $K = R(\alpha)$ for some $\alpha \in K$. Let $f$ be the minimal polynomial of $\alpha$ over $K$. Then $f$ is irreducible, but $\deg f = [K : R]$ is odd, so $f$ has a root in $R$. Therefore, $[K : R] = \deg f = 1$; that is, $K \cong R$. $\square$

**Lemma 15.** *The field $R(i)$ is the unique quadratic extension of $R$.*

*Proof.* Let $K/R$ be a quadratic extension. Since $\operatorname{char} R \neq 2$, we have $K \cong R(\sqrt{a})$ for some $a \in R$. Since $a$ cannot be a square in $R$, we know $-a$ must be one. Rescaling by $\sqrt{-a}$, we get $R(\sqrt{a}) \cong R(i)$. $\square$

**Lemma 16.** *There is no quadratic extension of $R(i)$.*

*Proof.* Since $\operatorname{char} R(i) \neq 2$, it suffices to show that every element of $R(i)$ is a square. Observe that every $x \in R$ is a square in $R(i)$: if $x$ is not a square in $R$, then $-x$ is, and so $x = (i\sqrt{-x})^2$. Further, since the negation of a sum of squares cannot be a square in $R$ (otherwise, dividng through and rearranging, $-1$ is a sum of squares), sums of squares in $R$ are squares in $R$. Now, fix $x = a + bi \in R(i)$ with $a, b \in R$. If $b = 0$, then $a = x$ is a square in $R(i)$. Otherwise, we have $x = (c + di)^2$, where

$$c = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \text{ and } d = \frac{b}{2c}.$$

Note that $d$ is well-defined because, if $c = 0$, then $-a = \sqrt{a^2 + b^2}$ and so $b = 0$. $\square$

**Theorem 17.** *The only finite extensions of $R$ are $R$ itself and $R(i)$.*

*Proof.* By separability, every finite extension of $R$ is contained in a finite Galois extension. Since $R(i)/R$ has no intermediate fields, it suffices to show the result for finite Galois extensions.

Let $K/R$ be a nontrivial Galois extension of degree $2^k \cdot a$, where $k \geq 0$ and $a \geq 1$ is odd. Applying the Galois correspondence to a Sylow 2-subgroup of $\mathrm{Gal}(K/R)$ yields an intermediate extension of degree $a$; by Lemma 14, we have $a = 1$ (and $k > 0$). If $k > 1$, iterating the last construction yields intermediate extensions $K/L/M/R$ with $[L : M] = [M : R] = 2$. By Lemma 15, $M \cong R(i)$, contradicting Lemma 16.$_\#$ Therefore $k = 1$ and (by Lemma 15) $K \cong R(i)$. □

**Corollary 18.** *The only algebraic extensions of $R$ are $R$ itself and $R(i)$.*

*Proof.* An infinite algebraic extension contains finite subextensions of arbitrarily large degree. □

**Corollary 19.** $\bar{R} = R(i)$.

*Proof.* □

The converse to Theorem 17 is much easier.

**Lemma 20.** *Let $F$ be a field in which $-1$ is not a square, and suppose every element of $F(i)$ is a square. Then sums of squares in $F$ are squares in $F$.*

*Proof.* Given $a, b \in F$, find $c, d \in F$ such that $a + bi = (c + di)^2$ in $F(i)$. Then $a = c^2 - d^2$ and $b = 2cd$, so $a^2 + b^2 = (c^2 + d^2)^2$ is a square in $F$. By induction, every sum of squares in $F$ is a square in $F$. □

In particular, $F$ as in the last lemma is real.

**Lemma 21.** *Suppose $R$ is a field whose only nontrivial finite extension is $R(i)$. Then $R$ is real closed.*

*Proof.* By Lemma 20, $R$ is real.

Take a non-square $a$ in $R$. Then $R(\sqrt{a}) \cong R(i)$. Suppose $i$ maps to $x + y\sqrt{a}$ for some $x, y \in R$; then $-1 = x^2 + ay^2 + 2xy\sqrt{a}$. Comparing coefficients, we get $-1 = x^2 + ay^2$. Since $-1$ is not a square, $y$ must be nonzero, and so $-a = (x/y)^2$ is a square.

Now, fix a nonlinear odd-degree polynomial $f \in R[X]$. Then $R[X]/(f)$ cannot be a field since $R$ has no nontrivial odd-degree extensions, and so $f$ must be reducible. We are done by induction on the degree. □

As before, let $R$ be a real closed field. Theorem 17 is a powerful tool for deriving more of its properties.

**Lemma 22.** *$R$ has no nontrivial real algebraic extensions.*

*Proof.* The field $R(i)$ is not real since $-1$ is a square in it. We are done by Corollary 18. □

**Corollary 23.** *$R$ has no nontrivial ordered algebraic extensions (with respect to the unique order).*

*Proof.* Since ordered fields are real, we are done by Lemma 22. □

The next property is a little less obvious.

**Lemma 24.** *The monic irreducible polynomials over $R[X]$ have form $X - c$ for some $c \in R$ or $(X - a)^2 + b^2$ for some $a, b \in R$ with $b \neq 0$.*

*Proof.* Let $f \in R[X]$ be monic and irreducible. The field $R_f = R[X]/(f)$ is a finite extension of $R$, so it is classified by Theorem 17. If $R_f \cong R$, then $\deg f = 1$, so $f = X - c$ for some $c \in R$. If $R_f \cong R(i)$, let the isomorphism be $\varphi$, and suppose $\varphi(X + (f)) = a + bi$ $(a, b \in R)$. Note that $b \neq 0$ since $\varphi^{-1}$ is constant on $R$. Rearranging, we see that $\varphi((X-a)^2 + b^2 + (f)) = 0$; that is, $(X-a)^2 + b^2 \in (f)$. Since this polynomial is monic and has the same degree as $f$, it must in fact be equal to $f$.

Conversely, linear polynomials over a domain are irreducible by degree, and reducible quadratics have a root. A root of $f = (X-a)^2 + b^2$ with $a, b \in R$ is an element $r \in R$ satisfying $(r-a)^2 = -b^2$. Since squares are non-negative, if $b \neq 0$ then $f$ must be irreducible. □

**Lemma 25.** *Polynomials over $R$ satisfy the intermediate value property (with respect to the unique order).*

*Proof.* We will prove that, for all $f \in R[X]$ and all $a, b \in R$ with $a \leq b$, if $f(a) \leq 0 \leq f(b)$, then there is some $c \in [a, b]$ such that $f(c) = 0$.

Fix $a, b \in R$ with $a \leq b$, and proceed by induction on $\deg f$. If $\deg f = 0$, then $f$ is constant and the result is clear. Otherwise, take a monic irreducible factor $g$ of $f$; then $g$ is classified by Lemma 24.

Observe that, if $g(a)$ and $g(b)$ are both positive, then $f/g$ satisfies the inductive hypothesis, and so has a root in $[a, b]$. If $g = (X-a)^2 + b^2$ with $a, b \in R$ and $b \neq 0$, then $g$ is everywhere positive. If $g = X - c$ with $c \in R$, then either $c \in [a, b]$ and $g(c) = 0$, or $c \notin [a, b]$ and $f(a)$ and $f(b)$ have the same sign. In this last case, either $f/g$ or $f/(-g)$ satsifies the inductive hypothesis. In all cases, $f$ has a root in $[a, b]$. □

In fact, the converses to Lemmas 22 and 25 also hold!

**Theorem 26.** *Let $R$ be an ordered field whose polynomials satisfy the intermediate value property. Then $R$ is real closed.*

*Proof.* We use Lemma 12. Let $a \in R$ be non-negative, and consider the polynomial $f = X^2 - a$. Then $f(0) = -a \leq 0$, but $f(a+1) = a^2 + a + 1 > 0$. By the intermediate value property, $f$ has a root in $R$, and so $a$ is a square in $R$.

Let $f$ be an odd-degree polynomial over $R$. Write $f = a_n X^n + \cdots + a_0$. We will show $f$ has a root in $R$. Replacing $f$ by $-f$ if necessary, we may assume $a_n > 0$. For $x > 1$, we compute

$$f(x) \geq x^{n-1}(a_n x - n \max_i |a_i|).$$

Therefore, when $x > \max\{1, n \max_i |a_i|/a_n\}$, $f(x) > 0$. A similar calculation shows that $f(x) < 0$ for sufficiently large negative values of $x$. We are done by the intermediate value property. □

**Theorem 27.** *Let $R$ be an ordered field with no nontrivial ordered algebraic extensions. Then $R$ is real closed.*

*Proof.* We use Lemma 12.

Let $a \in R$ be non-negative, and suppose $a$ is not a square. By Corollary 7, there is an ordering making the nontrivial extension $R(\sqrt{a})/R$ ordered.$_{\#}$ Therefore $a$ is a square in $R$.

By induction on degree, it suffices to show that irreducible odd-degree polynomials over $R$ are all linear. Let $f \in R[X]$ be such a polynomial, and consider the odd-degree field extension $R_f = R[X]/(f)$. By Lemma 8, there is an ordering making $R_f/R$ an ordered extension. Therefore $\deg f = [R_f : R] = 1$. □

**Corollary 28.** *Let $R$ be a real field with no nontrivial real algebraic extensions. Then $R$ is real closed.*

*Proof.* Since ordered fields are real, we are done by Theorem 27. □

Theorem 27 gives us a way to "construct" real closed fields.

**Lemma 29.** *An algebraically closed field of characteristic 0 has an index-2 real closed subfield.*

*Proof.* Let $C$ be an algebraically closed field of characteristic 0. Observe that the prime subfield $\mathbb{Q}$ can be ordered. Further, given an ordered subfield $F$ with $\bar{F} \neq C$, we can use Lemma 9 to adjoin an element transcendental over $F$, obtaining a strictly bigger ordered subfield.

Apply Zorn's lemma to obtain a maximal ordered subfield $R \subseteq C$; then $\bar{R} = C$. By Theorem 27, $R$ must be real closed. By Corollary 19, $C \cong R(i)$, and so $[C : R] = 2$. □

In summary, we have proved the following characterisations of real closed fields.

**Theorem 30.** *Let $R$ be a field. TFAE:*

1. *$R$ is real closed.*
2. *$\bar{R} = R(i)$ (and $-1$ is not a square in $R$).*
3. *$R$ is real, but has no nontrivial real algebraic extensions.*

*Proof.* □

**Theorem 31.** *Let $R$ be an ordered field. TFAE:*

1. *$R$ is real closed.*
2. *Polynomials over $R$ satisfy the intermediate value property.*
3. *$R$ is maximal with respect to ordered algebraic extensions.*

*Proof.* □

## 0.3 Real Closures

**Definition 32.** Let $F$ be an ordered field. A real closure of $F$ is a real closed ordered algebraic extension of $F$.

**Lemma 33.** *Let $F$ be an ordered field. Then $F$ has a real closure.*

*Proof.* Apply Zorn's lemma to ordered algebraic extensions of $F$. We are done by Theorem 27. □

Just like with the algebraic closure, it makes sense to talk of *the* real closure of an ordered field. Proving this uniqueness result requires a method of root-counting in real fields known as Sturm's theorem.

**Theorem 34** (Corollary to Sturm's Theorem)**.** *Let $F$ be an ordered field, and let $f$ be a polynomial over $F$. Then $f$ has the same number of roots in any real closure of $F$.*

*Proof.* TODO : decide on the generality of the statement of Sturm's Theorem □

**Lemma 35.** *Let $F$ be an ordered field with a real closure $R$, and let $K/F$ be a finite ordered extension. Then there is an $F$-homomorphism $K \to R$.*

*Proof.* By the primitive element theorem, $K = F(\alpha)$ for some $\alpha \in K$. Let $f$ be the minimal polynomial of $\alpha$ over $F$. Since $F$ has a root in $K$, it has a root in a real closure of $K$ (one exists by Lemma 33). By Theorem 34, $f$ has a root $\beta$ in $R$. Therefore define $\varphi : K \to R$ with $\varphi(\alpha) = \beta$. $\square$

**Lemma 36.** *Let $F$ be an ordered field with a real closure $R$, and let $K/F$ be a finite ordered extension. Then there is a unique order-preserving $F$-homomorphism $K \to R$.*

*Proof.* Fix a real closure $R'$ of $K$ (one exists by Lemma 33).

By the primitive element theorem, $K = F(\alpha)$ for some $\alpha \in K$. Let $f$ be the minimal polynomial of $\alpha$ over $F$, and let $\alpha_1 < \cdots < \alpha_m$ be the roots of $f$ in $R'$, with $\alpha = \alpha_k$. By Theorem 34, $f$ also has $m$ roots in $R$; let them be $\beta_1 < \cdots < \beta_m$. Since non-negative elements of $R'$ are squares, and $x_1, \ldots, x_{m-1} \in R'$ such that $\alpha_{j+1} - \alpha_j = x_j^2$, and let $L = K(\alpha_1, \ldots, \alpha_m, r, x_1, \ldots, x_{m-1}) \leq R'$. Now, suppose we have a $K$-homomorphism $\psi : L \to R$. Each $\psi(\alpha_j)$ is equal to a different $\beta_i$. Then $\psi(\alpha_{j+1}) - \psi(\alpha_j) = \psi(x_j)^2 \geq 0$, so $\psi(\alpha_1) < \cdots < \psi(\alpha_m)$, and so $\psi(\alpha_j) = \beta_j$ for all $j$.

By Lemma 35, there is in fact an $F$-homomorphism $\varphi : L \to R$. We will show that $\varphi$ is order-preserving. Indeed, fix a non-negative element $x \in L$. As before, find $r \in R'$ such that $x = r^2$, and let $M = L(r) \leq R'$. Apply Lemma 35 again to obtain an $L$-homomorphism $\psi : M \to R$; then $\varphi(x) = \psi(x) = \psi(r)^2 \geq 0$. Therefore $\varphi$ maps non-negative elements to non-negative elements, and so is order-preserving. Then $\varphi|_K$ is the map we want. Note that $\varphi(\alpha) = \beta_k$.

To see uniqueness, let $\tilde{\varphi} : K \to R$ be an order-preserving $F$-homomorphism; by existence, $\tilde{\varphi}$ extends to a an order-preserving $K$-homomorphism $\tilde{\psi} : L \to R$. Then $\tilde{\varphi}(\alpha) = \tilde{\psi}(\alpha_k) = \beta_k = \varphi(\alpha)$, and so $\tilde{\varphi} = \varphi$. $\square$

Taking $K = F$ above, we see that the order-embedding of a field into its real closure is unique.

**Theorem 37.** *Let $F$ be an ordered field. Then the real closure of $F$ is unique up to unique $F$-isomorphism.*

*Proof.* Let $R_1$ and $R_2$ be real closures of $F$. Applying Zorn's lemma to the set of ordered extensions intermediate between $R_1$ and $F$ having a unique order-preserving $F$-embedding into $R_2$, and using Lemma 36, we obtain an intermediate extension $R_1/K/F$ with no nontrivial finite ordered extensions and a unique order-preserving $F$-embedding $\varphi : K \to R_2$. If the ordered algebraic extension $R_2/\varphi(K)$ were nontrivial, then it would contain a nontrivial ordered finite extension, so $\varphi$ must be surjective (and so an $F$-isomorphism). In particular, $K \subseteq R_1$ is real closed; by maximality (Lemma 22), in fact $K = R_1$ and so $\varphi$ is an $F$-isomorphism between $R_1$ and $R_2$. $\square$

**Corollary 38.** *A real closed field has no nontrivial field automorphisms.*

*Proof.* Let $R$ be a real closed field. By Theorem 37, $R$ has no nontrivial order-preserving automorphisms. Since the ordering on $R$ is unique (by Lemma 13), every automorphism of $R$ must be order-preserving. $\square$

This uniqueness result is stronger than the one in the algebraically closed case: an algebraically closed field has many nontrivial automorphisms.

Uniqueness of algebraic closures allows us to classify ordered algebraic extensions.

**Lemma 39.** *Let $F$ be an ordered field with real closure $R$, and let $K/F$ be algebraic. Then field orderings on $K$ making $K/F$ ordered correspond to $F$-homomorphisms $K \to R$ via the order obtained by restriction from $R$.*

*Proof.* Fix an ordering on $K$ extending that on $F$, and let $K$ have real closure $R_K$ (exists by Lemma 33). Then $R_K/F$ is algebraic, so $R_K$ is a real closure of $F$. By Theorem 37, there is an $F$-isomorphism to $R_K \cong R$, and this induces an $F$-homomorphism $K \to R$. Restricting the order on $R$ to $K$ via this map recovers the original order on $K$ by construction.

Moreover, the inverse to order restriction constructed above is unique. Indeed, an inverse $\varphi : K \to R$ is order-preserving by definition, so it is an order-embedding from $K$ into its real closure. By Theorem 37, such a map is unique. □

## 0.4 The Artin-Schreier Theorem

We can go much further than Lemma 21. The following is a weak form of the Artin-Schreier theorem. Removing the condition on characteristic is possible, but requires some more involved algebra.

**Theorem 40.** *Let $R$ be a field with* $\operatorname{char} K \neq 2$*, and suppose $[\bar{R} : R] = 2$. Then $R$ is real closed.*

*Proof.* By Lemma **??**, it suffices to show that $\bar{R} \cong R(i)$.

Since $\operatorname{char} \bar{R} \neq 2$, we have $\bar{R} \cong R(\sqrt{a})$ for some non-square $a \in R$. Since $R(\sqrt{a})$ is algebraically closed, $\sqrt{a}$ is a square in $R(\sqrt{a})$; find $x, y \in R$ such that $\sqrt{a} = (x + y\sqrt{a})^2$. Expanding and comparing coefficients, $x^2 + y^2 a = 0$ and $2xy = 1$. Rearranging, $a = -(4x^4) = -1 \cdot (2x^2)^2$. Rescaling by $2x^2$, $R(i) \cong R(\sqrt{a}) = \bar{R}$. □

We can weaken the hypotheses even further. Here is the full Artin-Schreier theorem.

**Theorem 41** (Artin-Schreier Theorem)**.** *Let $R$ be a field, and suppose $\bar{R}$ is a finite extension of $R$. Then $R$ is real closed.*

*Proof.* TODO: this needs a lot more preliminaries eg Artin-Schreier theory, Kummer theory □

**Corollary 42.** *An algebraically closed field of nonzero characteristic has no finite-index subfields.*

*Proof.* Ordered fields have characteristic 0. □

**Theorem 43.** *The finite-index subfields of $\bar{\mathbb{Q}}$ are isomorphic copies of $\mathbb{Q}_{alg} = \bar{\mathbb{Q}} \cap \mathbb{R}$ indexed by* $\operatorname{Gal}(\mathbb{Q}_{alg}/\mathbb{Q}(i))$.

*Proof.* Since $\mathbb{Q}_{\mathrm{alg}}(i) = \bar{\mathbb{Q}}$, the field $\mathbb{Q}_{\mathrm{alg}}$ is a finite-index subfield.

By Theorem 41, any finite-index subfield has a unique ordering making it real closed. Let $R$ be a real closed subfield. Then the ordering on $R$ restricts to the ordering on $\mathbb{Q}$ (unique by Corollary 4). Since $R/\mathbb{Q}$ is algebraic, $R$ is a real closure of $\mathbb{Q}$. By Theorem 37, there is a unique $\mathbb{Q}$-isomorphism $\psi : \mathbb{Q}_{\mathrm{alg}} \cong R$. Since $\mathbb{Q}_{\mathrm{alg}}(i) = R(i) = \bar{\mathbb{Q}}$, $\psi$ extends uniquely to $\varphi \in \operatorname{Gal}(\mathbb{Q}_{alg}/\mathbb{Q}(i))$ by mapping $i \to i$. The subfield $R$ is then recovered from $\varphi \in \operatorname{Gal}(\mathbb{Q}_{alg}/\mathbb{Q}(i))$ by taking $\varphi^{-1}(\mathbb{Q}_{\mathrm{alg}})$. □

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